CURRICULUM MODELS IN MATHEMATICS

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CURRICULUM MODELS IN MATHEMATICS

by

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PREAMBLE

This report was commissioned by the Ministry of Education as part of the 1981 Provincial Assessment of Mathematics. Its purpose is to provide background information leading to identification of mathematics curriculum model appropriate to the needs of British Columbia.

A major goal of the 1981 Mathematics Assessment was to provide information for curriculum developers. The Assessment preceded a scheduled review of the mathematics curriculum which, in turn, will be followed by a revision. To provide a base from which to establish goals and priorities for the future, the Contract Team for the Assessment was required to produce a scholarly paper on the major mathematics models currently guiding mathematics programs throughout North America and elsewhere in the world. This paper identifies the theoretical foundations on which these models are built and the potential strengths and weaknesses inherent in each.

Sincere appreciation is expressed to David Robitaille and Michael Dirks of the University of British Columbia for the comprehensiveness and clarity with which they have handled the difficult task of summarizing the relevant literature. Their identification of factors which influence the curriculum and their determination of implications for a future British Columbia mathematics model will provide a valuable reference to curriculum decision-makers.

Joyce Matheson
Director
Learning Assessment Branch

PREFACE

The terms of reference for the 1981 Mathematics Assessment in British Columbia included a call for a paper dealing with different views of the structure and content of the mathematics curriculum. That paper, which appeared as Chapter 2 of the General Report of the assessment, was intended to serve both as a framework for the entire assessment and as a source document for the process of curriculum revision in mathematics in this province.

The paper published here is a somewhat modified and expanded version of the one which appeared in the General Report. A number of brief sections have been added, many minor changes have been made, and the list of references has been updated. A third version of the paper, one which differs only slightly from the present one, is being published in the journal, For the Learning of Mathematics.

Part I of the paper recalls the importance traditionally assigned to mathematics in the school curriculum and notes the lack of a strong rationale for that importance to be found in the literature. Part II contains a brief sketch of some of the major schools of thought on the nature of the discipline of mathematics and a discussion of their relationship to the subject as it is taught in schools. In Part III four large classes of factors which affect the school mathematics curriculum in significant ways are identified and discussed.

The first three sections are preliminary to an examination, in Part IV, of three models or paradigms for the mathematics curriculm and three curricula are then analyzed in the light of those models. Finally, in Part V, a number of implications of the three models for the process of curriculum revision are drawn.

Although the paper is rather lengthy, it is far too brief to treat all of the important issues involved in an adequate fashion. An extensive list of references has been included so that readers may pursue those topics which are of particular interest to them in greater detail.

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PART I - INTRODUCTION

Mathematics has always been a major component of the school curriculum. Mmari (1980) reminds us, for example, that Plato included mathematics as part of the training of the philosopher-king in The Republic. Although there are differences among countries regarding the specific mathematical topics that are included in the curriculum, in the proportion of students who study that content, and in the ways in which the content is taught, there is no disagreement regarding the basic fact of the matter. Mathematics is universally considered to be an important component of schooling.

A great deal has been written on the topics of goals for school mathematics and reasons for teaching mathematics. (See, for example, Ahlfors and others, 1962; Watson, 1971; Edwards, Nichols, and Sharpe, 1972; Braunfeld, Kaufman, and Haag, 1973; O'Brien, 1973; Bell, 1974; Evyatar, 1974; Hendrickson, 1974; Christiansen, 1975; Hershkowitz, Shami, and Rowan, 1975; Servais, 1975; Matthews, 1976; McNelis and Dunn, 1977). In the past decade Unesco has sponsored the publication of a number of reports entitled New Trends in Mathematics Teaching which deal with changes in the teaching of mathematics around the world. The most recent report, Volume IV, was published in 1979. Moreover, in 1978 Educational Studies in Mathematics published sixteen papers from different countries dealing with changes in the mathematics curriculum during the past twenty years.

In spite of the size of this body of literature, it is difficult to avoid concluding, as W.W. Sawyer (1948) does, that "the fact is that nobody knows why mathematics is taught in schools. Teaching mathematics is a custom, like shaking hands. We have got used to it. People cannot imagine schools without an arithmetic lesson." (page 8)

The allegation that no one knows why we teach the mathematics we do to so many students may be somewhat exaggerated, but it does contain a certain amount of truth. An examination of documents describing school mathematics curricula makes it appear that, in some cases, little thought has been given to fundamental issues such as the goals of the mathematics curriculum. Where goals have been identified it is often difficult to see how the content selected for inclusion in the curriculum is related to those goals. In certain instances it would seem that curricular goals are stated and then largely ignored. Content would appear to have been selected, as Sawyer says, on the basis of tradition or in response to the latest swing of the pendulum of curricular fashion.

Change in the mathematics curriculum is sometimes undertaken without sufficient prior consideration of goals and objectives. Such change seems pre-ordained to be transitory and not extremely influential. Successful adoption and implementation of a revised curriculum requires, as a prerequisite, careful weighing of the reasons for change and an in-depth evaluation of the goals of the curriculum. Alfred North Whitehead, the British philosopher, mathematician, and co-author with Bertrand Russell of the Principia Mathematica, addressed himself to the issue of change in the mathematics curriculum on the occasion of his presidential address to the Mathematical Association (Whitehead, 1916). In decrying the process by which such changes were introduced in a piecemeal fashion he said:

> This question of the degeneration of algebra into gibberish, both in word and in fact, affords a pathetic instance of the uselessness of reforming educational schedules without a clear conception of the attributes which you wish to evoke in the living minds of the children....
> You cannot put life into any schedule of general education unless you succeed in exhibiting its relation to some essential characteristic of all intelligent or emotional perception. (page 197)

More recently, in reporting the results of an observational study of the implementation of New Math curricula, Sarason (1971) commented as follows:

> The attempt to introduce a change into the school setting makes at least two assumptions: the change is desirable according to some set of values, and the intended outcomes are clear.... The new math...illustrates the problem of intended outcomes clearly.... Neither in the specific case we described nor in the general literature is it clear what outcomes were intended, whether or not there was a priority among outcomes, and what the relationship is between any outcome and the processes of change leading to it. (pages 62-63)

Regarding the process of curricular change, Sarason asserts that "intended consequences are rarely stated clearly, if at all, and as a result, a means to a goal becomes the goal itself, or it becomes the misleading criterion for judging change." (page 48)

The formulation of goals, the construction of a mathematics curriculum, and the successful implementation of that curriculum require an understanding of the nature of mathematics itself, of school mathematics, and of the interaction between the two. (See Figure 1) This interaction may be seen as consisting of a number of factors which operate on the body of mathematics to select and restructure the content deemed to be most appropriate for the school curriculum.

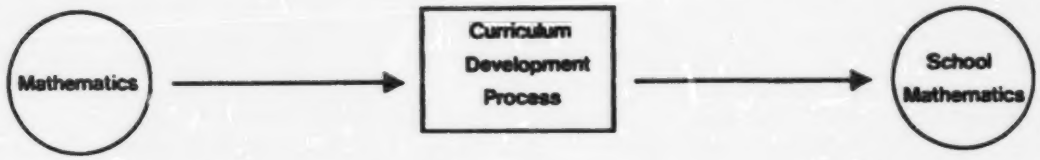

Figure 1. Origins of the school mathematics curriculum.

In the remainder of this paper each of the three components is examined in some detail. The nature of mathematics is discussed first. This is followed by an examination of four forces which affect and influence the curriculum development process. Then three models or paradigms for the mathematics curriculum are described as outgrowths of the two earlier phases. Finally, some comments are made concerning the implications of such an analysis for the process of curriculum revision.

PART II - THE NATURE OF MATHEMATICS

In a footnote to a paper dealing with aims of mathematics education, Watson (1971) says: "It has been conjectured that there exists an n_0 such that for $n > n_0$ any set of n mathematicians will contain at least one pair who disagree on the definition of mathematics. It is believed that $n_0=2$." (page 106)[1] As in the case of goals for the teaching of mathematics, there is a substantial body of literature dealing with the question of the nature of mathematics.

In this literature, there is virtual unanimity that mathematics is not monolithic, but multi-faceted. Beltzner, Coleman and Edwards (1976), in their review of the state of the mathematical sciences[2] in Canada, characterize the field in three ways: (1) as a powerful means for analyzing experience, (2) as a cultural resource, and (3) as an important language essential for modern communication. Furthermore, they propose as "the chief and overriding aim for the teaching of mathematics in the schools the widest possible dissemination of an understanding of what mathematics is and what it is not." (page 117) (emphasis in the original) They go on to say that, to their knowledge, such an aim has never been made explicit by any school system.

Many categorizations and partitionings of mathematics have been made over the years. It is commonplace, for example, to distinguish between pure and applied mathematics.

The historian of science, Thomas S. Kuhn (1977), provides some important insights confirming both the relatively recent emergence of the pure-applied distinction, and the divergence

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[1] A number of readers of an earlier draft of this paper have suggested strengthening Watson's conjecture by changing the condition to $n \geq n_0$. This new conjecture would appear to be founded upon a considerable amount of first-hand experience in the field.

[2] Steen (1978a) observes that "in the past quarter century the world of mathematics has evolved from a single discipline to a cluster of intertwined subjects now usually termed 'mathematical sciences'." (page 7) He lists as examples the fields of number theory, mathematical logic, differential topology, algebraic geometry, operations research, statistics, computer science, combinatorics, mathematical programming, mathematical economics, mathematical biology, mathematical psychology, mathematical linguistics, and cliometrics.

of opinion internationally regarding the exact nature of this distinction. He writes:

> A(n).....important source of change during the nineteenth century was a gradual shift in the perceived identity of mathematics. Until perhaps the middle of the century such topics as celestial mechanics, hydrodynamics, elasticity, and the vibrations of continuous and discontinuous media were at the center of professional mathematical research. Seventy-five years later, they had become "applied mathematics", a concern separate from and usually of lower status than the more abstract questions of "pure mathematics" which had become central to the discipline.... [This separation] occurred in different ways and at different rates in different countries. (pages 60-61)

Historically, there has been considerable and, at times, heated discussion about the nature of mathematics and, inevitably, about the type of content most appropriate to school mathematics. An examination of some of the details of such discussions will help clarify certain aspects of the curriculum models to be presented later.

Pure Mathematics

Much has been written (Eves and Newsom, 1963; Benacerraf and Putnam, 1964; Nagel and Newman, 1968; Wilder, 1968; Barrett, 1979) about attempts around the turn of the century to secure the foundations of mathematics after the surge of research activity in the preceding two centuries. The details of Russell's logicism, Brouwer's intuitionism, and Hilbert's formalism need not be examined here[3], but some consideration must be given them if only because it has been argued elsewhere (Kline, 1977; Barrett, 1979) that the goals of logicism and formalism, in particular, have had an impact on both the orientation of research in mathematics and on the mathematics curriculum of the schools.

According to Russell (Moritz, 1958), "Pure mathematics is the class of all propositions of the form 'p implies q', where

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[3]Hanna (1981) describes these three philosophical orientations in some detail in her doctoral dissertation in which she examines the status of rigorous proof and the axiomatic method in contemporary textbooks for school mathematics.

p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants." Russell goes on to say that "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." (page 7)

The formalist program of David Hilbert (a German of the Göttingen school as were Gauss, Riemann, Klein, and later Courant) has, even more than the logicist one, had an impact on the school mathematics curriculum. Hilbert himself was an eminent mathematician who worked in a number of different fields and with varying degrees of rigor. He is credited (Eves and Newsom, 1963) on the one hand with having "sharpened the mathematical method from the material axiomatics of Euclid to the formal axiomatics of the present day," (page 142) and on the other hand with having produced a classic intuitive exposition of geometry in Geometry and the Imagination with Cohn-Vossen.

The basis for Hilbert's formalist position was his desire to provide a consistent, finite, axiomatic foundation for all of classical analysis (Wang, 1978). In Hilbert's view "mathematical expressions are regarded simply as empty signs. The postulates and theorems constructed from the system of signs ... are simply sequences of meaningless marks which are combined in strict agreement with explicitly stated rules." (Nagel and Newman, 1968, page 224). Von Neumann (1964) described the formalist approach as follows:

> The leading idea of Hilbert's theory of proof is that, even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians and which consists basically in constructing successively certain combinations of primitive symbols which are considered "correct" or "proved". (page 50)

Hilbert's goal of establishing the absolute consistency of a deductive system was shown by Gödel, in 1931, to be unattainable for any system as complex as that of the arithmetic of natural numbers. In his Incompleteness Theorem, Gödel demonstrated that any "suitably strong" system will contain statements whose truth or falsity is undecidable within that system. (Hofstadter, 1979)

According to Barrett (1979), mathematics research in America has been dominated by the goals of formalism. Felix Browder and Saunders Mac Lane (1978), both eminent American

mathematicians, draw a careful distinction between Hilbert's formalist program and the research paradigm which grew out of it.

> In its most vulgar form (and it is the vulgar forms of sophisticated intellectual doctrines that tend to sweep across the intellectual landscape), the formalist doctrine was taken to say that mathematics consists simply of the formal manipulation of uninterpreted symbols, or of reasoning by formal deduction (itself reduced simply to symbolic manipulation) from any assumptions whatever as long as they could be presented in an explicitly symbolic form. Taken in this form (and it is clear from his explicit statements that Hilbert would have found this form of the thesis horrifying), the vulgar formalist doctrine argued against even the possibility of any objective content for any part of mathematics. It even seemed to argue against the significance and the content of the historically conditioned mathematical fields, as well as against their intuitions and central problems. In the context of applying mathematics to the analysis of the phenomena of the natural world, it makes any significant application a miracle on principle and a triumph of will over content. (page 344)

Browder and Mac Lane then go on to argue that the effect of what they term "vulgar formalism" has been felt not only in the mathematical culture but even more so in some scientific fields which make use of mathematics. They contend that:

> While the vulgar formalist attitude towards mathematics has never achieved a totally dominant influence upon mathematical research even when unchallenged as an ideology, the very similar instrumentalist viewpoint of the contemporary theoretical physicist towards mathematics became dominant in physics. (page 345)

Although formalism and abstraction are certainly not equivalent terms, the abstract orientation of much of the mathematical research on this continent and elsewhere during this century has reinforced the formalist view of the nature of mathematics. As Marshall Stone (Griffiths & Howson, 1974) has noted:

> When we stop to compare the mathematics of today with mathematics as it was at the close

> of the nineteenth century, we may well be amazed to note how rapidly our mathematical knowledge has grown in quantity and complexity, but we should also not fail to observe how closely this development has been involved with an emphasis on abstraction and an increasing concern with the perception and analysis of broad mathematical patterns. Indeed, upon closer examination we see that this new orientation, made possible only by the divorce of mathematics from its applications, has been the true source of its tremendous growth during the present century. (page 120)

While not disagreeing with Stone as to the facts of the matter, the following quotation from The Thirty-Second Yearbook of the National Council of Teachers of Mathematics (NCTM) (Jones, 1970) presents a somewhat less-glowing picture:

> The tendency for American mathematical research to be very pure, and in the general area of the foundations of mathematics rather than in applied mathematics, continued into the twentieth century, becoming a real national handicap at the time of World War II. (page 30)

One of the clearest indications of the impact of abstract formalistic thinking on school mathematics in America may be found in the following remarks by the late E. G. Begle (1979), the mathematician-turned-educator who headed the extremely influential School Mathematics Study Group (SMSG) in the U. S. A. He said, "I consider mathematics to be a set of interrelated, abstract, symbolic systems." (page 1) Two pages later he stated that, "We are dealing with an abstract (the symbols, so far, are meaningless), symbolic (we have nothing but symbols and strings of symbols) system." (page 3) (emphasis in the original) Although Browder and Mac Lane (1978) feel that the influence of formalism on professional mathematical activity is in decline, they also maintain that "vulgar formalism has been spread on a much more explicit level and to a much wider public than it ever reached before through the formalist thrust of a large part of the new curricula in the elementary and secondary schools." (page 345)

Hersh (1979), a critic of formalism and of the goal of providing "secure" foundations for mathematics in general, states that there is a definite connection between this orientation and the reality of school mathematics, not only in terms of specific content but also in terms of presentation and motivation of the subject. He contends:

> The last half-century or so has seen the rise of formalism as the most frequently advocated point of view in mathematical philosophy. In this same period, the dominant style of exposition in mathematical journals, and even in texts and treatises, has been to insist on precise details of definitions and proofs, but to exclude or minimize discussion of why a problem is interesting, or why a particular method of proof is used.... One's conception of what mathematics is affects one's conception of how it should be presented.... Another example is the importation, during the '60's, of set-theoretic notation and axiomatics into the high-school curriculum. This was not an inexplicable aberration, as its critics sometimes seem to imagine. It was a predictable consequence of the philosophical doctrine that reduces all mathematics to axiomatic systems expressed in set-theoretic language. (page 33) (emphasis in the original)

Hersh goes on to conclude that

> (1) The unspoken assumption in all foundationist viewpoints is that mathematics must be a source of indubitable truth.
> (2) The actual experience of all schools--and the actual daily experience of mathematicians--shows that mathematical truth, like other kinds of truth, is fallible and corrigible. (page 43)

Such views are similar to those formulated by Imre Lakatos, whose mentors were Karl Popper and George Polya. The recent focus on problem-solving within the mathematics education community (NCTM, 1980) provides special impetus for critically considering Lakatos' stance concerning the nature of the discipline. In his book, Proofs and Refutations: The Logic of Mathematical Discovery, Lakatos (1976) asserts that:

> Formalism denies the status of mathematics to most of what has been commonly understood to be mathematics, and can say nothing about its growth. None of the "creative" periods and hardly any of the "critical" periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impurities of earthly uncertainty. (page 2)

In Lakatos' opinion, "informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations." (page 5)

Applied Mathematics

In contrast to the formalist view of mathematics as pure is that held by those who see the subject as being closely linked to its applications. Interestingly enough, so-called applied mathematicians have frequently refused to adopt an either-or position on the pure versus applied distinction, preferring instead to promote a compromise between the two extremes. Thus, Richard Courant (1941) in the preface to his classic text, What is Mathematics?, says:

> "Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science.
>
> "Without doubt, all mathematical development has its psychological roots in more or less practical requirements. But once started under the pressure of necessary applications, it inevitably gains momentum in itself and transcends the confines of immediate utility. This trend from applied to theoretical science appears in ancient history as well as in many contributions to modern mathematics by engineers and physicists....
>
> "However, while the theoretical and postulational tendency of Greek mathematics remains one of its important characteristics and has exercised an enormous influence, it cannot be emphasized too strongly that application and connection with physical reality played just as important a part in the mathematics of antiquity, and that a manner of presentation less rigid than Euclid's was very often preferred." (pages xv-xvi)

He cautions his readers about the danger of over-emphasizing the deductive character of mathematics.

> If the crystallized deductive form is the goal, intuition and construction are at least the driving forces. A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules, and syllogisms, without motive or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value. (page xvii)

Similar opinions have been voiced in other countries such as the Soviet Union (Aleksandrov, Kolmogorov, Lavrent'ev, 1963) and the United Kingdom (Griffiths and Howson, 1974).

Henry Pollak (1979, page 233), in a paper discussing the interaction between mathematics and other school subjects, presents four definitions of applied mathematics.

(1) "Applied mathematics means classical applied mathematics," which includes the various branches of analysis as well as algebra, trigonometry, and geometry from the secondary school curriculum.
(2) "Applied mathematics means all mathematics that has significant practical application." This would include all of the topics typically found in the secondary school curriculum as well as probability, statistics, linear algebra, and computer science.
(3) "Applied mathematics means beginning with a situation in some other field or in real life," constructing and working within a mathematical model of the situation, and then applying the results to the original situation.
(4) "Applied mathematics means what people who apply mathematics in their livelihood actually do." This definition is similar to the one given in (3).

He then goes on to discuss, in some detail, each of these definitions and to show the implications of each for the school curriculum.

Pollak's third and fourth definitions emphasize process rather than content, and a key implication for school mathematics emanating from these statements is that the process of model building should be emphasized rather than the applications themselves. Howson (1978), for example, is quite critical of some of the material produced by the Sixth Form Mathematics Project (an applied, upper-secondary level project) for merely putting narrow problems in an applied setting and avoiding the model building process. (page 207)

The term "mathematization" has been used to describe an approach to the teaching and learning of mathematics based upon the skills of model-building, and John Trivett of Simon Fraser University and David Wheeler of Concordia have been among the Canadian proponents of such an instructional approach. Beltzner et al. (1976) quote Wheeler approvingly in this regard, and then give the following admonition:

> The absorption and retention of "facts" should not be the main object of the teaching-learning process--but rather an important by-product. The main aim should be that of exploiting, and extending the ability to "mathematize" which is inherent in all thinking individuals--by encouraging the student to exult in the "algebraic" character of mental functioning. (page 119)

Mathematics as a Creative Art

Mathematicians, whether pure or applied, have often written and spoken about their field in terms one usually associates with the fine arts rather than the sciences. They see mathematics as a realm of endeavor characterized by insight, creativity, and beauty. One of the major arguments to be found in the literature concerning the reasons for studying mathematics is the inherent beauty of the subject, its structures, and its patterns.

G.H. Hardy (1967), the British mathematician, put it the following way:

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with <u>ideas</u>. . . .
> The mathematician's patterns, like the painter's or the poet's, must be <u>beautiful</u>; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is

> the first test: there is no permanent place in the world for ugly mathematics. (pages 84-85) (emphasis in the original)

Hardy's mathematics was pure mathematics and he took great delight in that fact. He said, "I have never done anything 'useful'. No discovery of mine has made or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." (page 150) He did say, however, "I have added something to knowledge...which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them." (page 151)

The Hungarian-born American mathematician, Paul Halmos (1968) described his art in the following way:

> For the professional pure mathematician, mathematics is the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof. Simplicity, intricacy, and above all, logical analysis are the hallmark of mathematics. (page 380)

and later:

> It is, I think, undeniable that a great part of mathematics was born, and lives in respect and admiration, for no other reason than that it is interesting--it is interesting in itself.... Don't all of us feel the irresistible pull of the puzzle? Is there really something wrong with saying that mathematics is a glorious creation of the human spirit and deserves to live even in the absence of any practical application? (page 386)

Finally, he says:

> I hope just the same, that I've shown you that there is a subject called mathematics... and that that subject is a creative art. It is a creative art because mathematicians create beautiful new concepts; it is a creative art because mathematicians live, act, and think like artists; and it is a creative art because mathematicians regard it so. (page 389)

Applied mathematicians have also spoken about this issue. Morris Kline (1962) is an applied mathematician whose greatest technical contributions to mathematics have been in the area of the mathematical behavior of radio waves. He has also been a harsh critic of what he perceives as being an extreme overemphasis on formalism and abstraction in mathematics research and teaching. He, like Courant, rejects the pure-applied distinction and, regarding the artistic side of mathematics, observes:

> Mathematics offers artistic outlets not only in the creation of theorems and proofs, but in the expression of its material. A painter may have a great theme but he must also present it most effectively. The same in true in mathematics. The symbolism can be employed neatly and suggestively just as words are used in poetry. (page 671)

Later, he continues:

> Perhaps the best reason for regarding mathematics as an art is not so much that it affords an outlet for creative activity as that it provides spiritual values. It puts man in touch with the highest aspirations and loftiest goals. It offers intellectual delight and the exaltation of resolving the mysteries of the universe. (page 671)

Summary

The purpose of this section has been to examine several conceptions of the nature of mathematics, since it has been suggested (Otte, 1979) that the way in which mathematics is viewed by members of the mathematics community has a profound effect on the school curriculum as conceived and as realized. The categories discussed here were chosen more for the sake of convenience than out of any strong feeling for their superiority over some other categorization. Differing approaches to such a discussion of the nature of mathematics have been taken by Browder (1976) and Confrey (1980), to name only two. What seems clear is that some sort of analysis and discussion of the nature of mathematics is a prerequisite to the process of curriculum development.

The current literature in mathematics education contains increasingly frequent references to the ideas of Imre Lakatos, whose work was mentioned earlier. For example, Bill Higginson (1980) of Queen's University contends that "answers to some critical problems of long standing in mathematics education can be found in aspects of the theories of Karl Popper, Imre

Lakatos and Jean Piaget." (page 6) Some possible implications of Lakatosian ideas for school mathematics have been suggested by Hersh (1979). He contends,

> The criticism of formalism in the high schools has been primarily on pedagogic grounds: "This is the wrong thing to teach, or the wrong way to teach." But all such arguments are inconclusive if they leave unquestioned the dogma that real mathematics <u>is</u> precisely formal derivations from formally stated axioms. If this philosophical dogma goes unchallenged, the critic of formalism in the schools appears to be advocating a compromise in quality: he is a sort of pedagogic opportunist, who wants to offer the student less than the "real thing". The issue, then, is not, What is the best way to teach? but, What is mathematics really all about? To discredit formalism in pedagogy, one must challenge its philosophical base: the formalist picture of the nature of mathematics. Controversies about high-school teaching cannot be resolved without confronting problems about the nature of mathematics. (page 33-34) (emphasis in the original)

Whether Hersh has over-stated the case or whether as Agassi (1980) has suggested, a Lakatosian revolution is imminent, is far from clear. What is clear is that the prevalent perceptions of the nature of mathematics are an important determinant of the school mathematics curriculum.

PART III - FACTORS INFLUENCING THE CURRICULUM DEVELOPMENT PROCESS

Earlier, the curriculum development process was portrayed as a mediator between the domain of mathematics and that of school mathematics. The function of the curriculum development process is to restructure mathematics into a form appropriate for the school curriculum. This restructuring process is influenced by a number of factors, each of which affects the development of the mathematics curriculum in a specific way. Four such factors will be discussed here.[4]

Sociological Factors

Both the content and the methodology of school mathematics are influenced by sociological factors beyond the control of the school and by the nature of the school setting itself. As Bauersfeld (1980) has observed:

> Teaching and learning mathematics is realized in institutions which the society has set up explicitly to produce shared meanings among their members. Institutions are represented and reproduced through their members and that is why they have characteristic impacts on human interactions inside of the institutional.[sic] They constitute norms and roles; they develop rituals in actions and in meanings; they tend to seclusion and self-sufficiency; and they even produce their own content--in this case, school mathematics. (pages 35-36) (emphasis in the original)

The past twenty years or so have provided a number of examples of the influence of sociological factors on the mathematics curriculum of the schools. During the New Math era, for example, mathematics was considered by many as requiring increased attention and prominence in the schools. In the United States, all of the physical sciences as well as mathematics benefitted from an enormous influx of government and private foundation funds during the late 1950s and 1960s, in part as a result of the successful launching of the first unmanned earth satellite by the Soviet Union in 1957.

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[4]Howson, Keitel, and Kilpatrick (1981) offer a somewhat different categorization in Chapter 4 of their book, Curriculum Development in Mathematics, than that given here.

The picture in the United States regarding public support for mathematics and the physical sciences has changed dramatically in the last two decades. As the report of the National Advisory Committee on Mathematical Education (1975) makes clear, the importance of mathematics and the sciences has declined in public opinion, and students' interests are being directed elsewhere. As a result, financial support for curriculum development and research in mathematics is not nearly as strong as it was.

More recently we have witnessed, on the one hand, the rise of "accountability" in the schools as a response to public pressure about the high costs of schooling; and, on the other hand, we have seen disquiet about claims that standards are falling. The back-to-basics movement[5], minimum competency testing, and the growth of provincial, state, and national assessment programs may all be viewed as responses by school systems to demands from the society which they serve.

Teachers are members of the society in whose schools they teach, and therefore are in a somewhat ambiguous position regarding innovations in teaching practice and materials. In House's (1979) opinion, many curricular innovations have failed because they were subverted by teachers and never implemented. The innovators frequently failed to account for the critical role of classroom teachers and neglected to involve them in the development process. For example, there is mounting evidence (Brandt and others, 1979; Fey, 1979; Robitaille, 1981a) that the New Math was never fully implemented in schools[6], and that teachers have not responded as positively or as enthusiastically as might have been expected to the adoption of the metric system or to the use of calculators and computers in schools (Robitaille and Sherrill, 1977; Robitaille, 1981b).

It has also been argued that innovations which stressed an inquiry approach or learning by discovery frequently had little impact because of their lack of fit with a rule-obedience function of school mathematics. Easley (1979), for example, contends that:

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[5]This movement is a world-wide phenomenon. According to Kolyagin, Lukankin, and Oganesyan (1980, page 74) in the Soviet Union it is known as the "return to Kiselev" movement.

[6]Beltzner et al. (1976) consider the New Math movement in Canada to have failed because, in their opinion, only the appearance of New Math was incorporated in most textbooks and those textbooks "attempt to teach jargon rather than ideas." (page 122)

> Some teachers might even admit to using arithmetic as a kind of moral training in the sense that students can be taught to be neat and orderly, to be responsible for doing their work on time and getting it turned in in proper form, and being responsible for their own work and not getting help from other people. These are really moral values for which arithmetic is seen by many teachers to be a suitable instrument, not the only instrument but one very suitable instrument. (page 9) (emphasis in the original)

Stake and Easley (1978) directed a project sponsored by the National Science Foundation which was designed to depict the teaching of mathematics and science in the United States. In summarizing their eleven case studies they speculate that:

> Subject-matter knowledge, as an end in itself (a common assumption of the academic community), got transformed in the school into a means of meeting the socialization demands of the school. It was clear to us that the school had a set of social norms (ways students were supposed to behave) which conflicted with the norms teachers were taught to espouse in teacher training courses. (page 16:5)

Researchers such as Waller (1965) and Lortie (1975) have attempted to portray the professional milieu of teachers from the sociologist's perspective. Lortie, for example, identifies "conservatism, individualism and presentism (as)...significant components in the ethos of American classroom teachers." (page 212) Individualism here refers in part to the fact that in the traditional school setting teachers have relatively few contacts with other adults and are seen as individually influencing students rather than as part of a coordinated system of influence. If this view is close to the reality of school life it poses problems for any curriculum implementation strategy based on a top-down approach and an assumption of consensus in educational goals throughout the different levels of the school system.

Sociological research into teaching offers useful insights but is as yet in an embryonic stage. Otte (1979) offers support for this opinion in a discussion of teacher education.

> The broad concept of practice has certainly added substantially to discussions of the components of mathematics teachers' activities;

> however, no substantiated analysis of school teaching and of its environmental conditions, no analysis of the daily activities of teachers and of the freedom of action they enjoy in their everyday work, have been produced so far. (page 112)

Although it has not been done here, it would certainly be possible to discuss the impact of sociological factors on the curriculum development process in a broader context. It seems reasonable to presume, for example, that the type of social and political system operating in a given place at a given time influences not only the educational system in general but, in particular, the mathematics curriculum. As Howson (1980) has observed:

> It is usual in mathematical periodicals--even those devoted to education--to fight shy of political issues. For example, curricula and school organisational patterns are presented and discussed in an aseptic, non-political manner. Yet mathematics education in any country cannot be divorced from politics, and we delude ourselves if we believe that this is not so. (page 285)

Howson continues with a critique of Swetz's (1978) formulation and analysis of the phenomenon of "socialist mathematics education". Such cross-political comparisons are beyond the scope of the present paper and, as can be inferred from Howson's paper, much careful analysis remains to be done before any meaningful conclusions can be drawn in this area. Nonetheless, it seems clear that any program of curriculum revision which seeks to incorporate aspects of an apparently successful innovation from a different jurisdiction must include an analysis of the social and political differences between the two places.[7]

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[7]MacDonald (1977), for example, has compared the relative influence of American and British approaches to the New Math on the Australian curriculum. In his opinion, "The upshot was that the transition to new mathematics programs in many parts of this country combined some of the worst features of both the U. K. and the U. S. experiences!" (page 56)

Psychological Factors

Griffiths and Howson (1974) have stated that in Great Britain psychological theories have not had a very great impact on the mathematics curriculum, especially at the secondary school level. They do say, however, that among the exceptions to this general rule are the materials produced by the Nuffield Project, the work of Her Majesty's Inspector Edith Biggs, and, to a lesser extent, that of Richard Skemp.

In North America, on the other hand, the mathematics curriculum and the teaching of mathematics have both been fairly heavily influenced by changing beliefs and theories about the ways in which children learn and what they are capable of learning at various age levels. Although some such beliefs and theories have been short-lived, a few have demonstrated remarkable durability and resistance to change.

A great many educators still subscribe to a form of the theories of mental discipline and transfer of training as they apply to mathematics. They believe that subjects like mathematics help teach students to "think logically" both in mathematical and non-mathematical situations. In a study conducted among teachers of secondary school mathematics in British Columbia (Robitaille, 1973), for example, it was found that when teachers were asked to rank some twenty objectives for the teaching of geometry, the five objectives which were ranked highest had nothing to do with geometry itself, or with mathematics for that matter. All of them were either mental discipline or transfer objectives. The belief, however ill-founded, that the study of Euclidean synthetic geometry will assist in the development of students' ability to think logically, has persisted in spite of results such as those reported by Fawcett (1938) and, more recently, Williams (1980).

Behaviorist theories of learning, from those of E. L. Thorndike to Robert Gagné, have also had a considerable and lasting influence on the teaching of mathematics. Thorndike's laws of exercise, effect, and readiness were based upon his rejection of a logical, axiomatic presentation of arithmetic. Critics of his connectionist psychology have asserted that he rejected meaning in mathematics along with axiomatics (Jones, 1970). Thorndike himself (1921) admonished his readers that:

> Arithmetic makes a very strong appeal to two potent interests--the interest in mental activity and the interest in achievement. Many children like arithmetic in the same way and for much the same reasons that they like puzzles, riddles, checkers, chess, and other

> intellectual games. Almost all children like to have their tasks definite so that they can know what they have to do and when it is done, and enjoy the sense of action, achievement, and mastery. (page 14)

He encouraged teachers to "use arithmetical games, races, matches, and the like ... when such games, races, and the like are just as instructive as mere drill for drill's sake." (page 28) While it is clear that Thorndike's formulation of bonding lost credibility in the 1930s, it must be recognized that many teacher- or commercially-produced games and puzzle worksheets are based upon a Thorndikean view of learning.[8] Smith (1976), in discussing the impact of Thorndike's work, says that "by and large [Thorndike's] conception of how learning occurs has permeated our thinking about teaching to such an extent that we are often unaware that we are persuaded by his views." (page 25)

More recently, the construction of so-called task analyses and behavioral objectives has enjoyed wide popularity in North America. The Dutch mathematician, Hans Freudenthal (1978), decries this "atomisation", to use his terminology, in the strongest terms pointing out that this practice has not taken such a firm hold in other countries.

During the New Math era the work of Jerome Bruner (1963) was extremely influential in the educational literature, if not in the typical classroom. His declarations to the effect that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development," (page 33) and that the best way to learn mathematics[9] is by behaving like a mathematician, were hallmarks of that period. At the primary school level, the developmental theory of Jean Piaget continues to exert an important influence on the teaching of mathematics in North America and elsewhere.

Piaget's (1973) conclusion that "there exists, as a function of the development of intelligence as a whole, a spontaneous and gradual construction of elementary logico-

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[8]Just as Browder and Mac Lane (1978) distinguished between Hilbert's view of formalism and its "vulgar" derivative, one might also distinguish between Thorndike's actual formulations and their vulgar derivatives.

[9]Bruner's comment was actually made about physics, not mathematics.

mathematical structures and that these 'natural' ... structures are much closer to those being used in 'modern' mathematics than to those being used in traditional mathematics" (page 79) has sometimes been used, along with Bruner's declaration, to justify the inclusion of modern mathematics in final, finished form in the school curriculum. However, Piaget himself states:

> With modern mathematics...the teacher is often tempted to present far too early notions and operations in a framework that is already very formal. In this case the procedure that would seem indispensable would be to take as the starting point the qualitative concrete levels: in other words, the representations or models used should correspond to the natural logic of the levels of the pupils in question, and formalisation should be kept for a later moment. (pages 86-87)

It should be noted in this connection that the Harvard Mathematics Project conducted by Bruner and Dienes (Dienes, 1963) relied heavily on the use of concrete apparatus although one of their objectives certainly was to accelerate the process of formalization.

Reactions to these theories and beliefs has varied both between and within countries. For example, Jim Fey (1978) of the University of Maryland criticized many of the American New Math programs for the elementary school as having been "conceived in a naive optimism that young children could achieve far more than had ever been expected of them." (page 341) On the other hand, Max Bell (1980) of the University of Chicago, in an address to the annual meeting of the NCTM, said that one of the major problems facing mathematics educators in the 1980s was a "pervasive pessimism about young children's mathematics abilities." (page 17-1)

Although a great deal of research has been done on the topic of how children learn mathematics, "progress has been slow and our knowledge of mathematical learning processes meagre." (Chapman, 1972, page 153) Many of the most fundamental questions remain only partially answered, and the list of issues or questions which have been answered definitively is all too short.

Pedagogical Factors

The methods and materials used by teachers of mathematics are important determinants of the mathematics curriculum as it is attained by students. So too are the qualifications, both

academic and professional, of the teachers themselves.

At the elementary school level in many countries, teachers have little or no academic preparation in mathematics beyond the secondary school level (OEEC, 1961). In British Columbia, for example, approximately 15% of the elementary school teachers surveyed in 1981 (Robitaille, 1981b) indicated that they had no post-secondary training in mathematics.

Elementary teachers tend to be generalists while their secondary school counterparts are more likely to be subject matter specialists. The resulting dichotomy between "child-orientation versus subject matter-orientation" has, in Otte's terms (1979, page 126), been a source of antagonism in discussions of the pedagogy of mathematics. The pros and cons of permitting only specialists to teach mathematics have been debated for many years, but no resolution of this argument appears imminent.

The type of training which universities provide for prospective teachers of mathematics is also an important issue. Reporting on the PRIME-80 conference which was sponsored by the Mathematical Association of America and which was intended to identify directions for the university-level training of future mathematicians and teachers of mathematics, Steen (1978b) states rather ominously that:

> Moreover, 60% of undergraduate mathematics enrollments are now in applied areas, with the remaining 40% split evenly between required and elective courses. Only those students preparing for careers as high school teachers are continuing to take the traditional courses - because they are required to by certification regulations. This situation ... is doubly dangerous: not only does it mean that future high school teachers may be ill-prepared to cope with their students' demands for new applications, but the commitment to teaching disqualifies them from other mathematics-related jobs that now universally require majors with an applied concentration. (page 172)

The subject matter content of the curriculum and the nature of the population of students at a given grade level who are studying mathematics vary considerably from place to place, particularly at the secondary school level. Thus, in some European countries, a substantial portion of the last year of secondary school mathematics is devoted to the study of calculus. In such countries, it is often the case that only a small proportion of the student population is permitted to continue their studies of mathematics beyond the minimum com-

pulsory level. In other words, these countries have formal "streaming" or "tracking" programs[10] whereby students who have been identified as highly capable in a given subject area are challenged to explore those areas in some depth. In England, for example, students in the sixth form--the two final, non-compulsory years of secondary school--specialize in two or three subjects. Such features of national school systems are not only part of the curriculum in the broad sense of the term but influence the content of the curriculum as well.

In both Canada and the United States, a larger proportion of students studies mathematics beyond the compulsory level than in many countries. In most school jurisdictions in North America however, there is some sort of provision made for talented students to specialize or to complete the regular courses in an accelerated fashion and then follow a program such as Advanced Placement, through which secondary school students can gain up to one-year's university credit in calculus. In British Columbia, although a few elective courses have been added to the secondary school mathematics curriculum, no program of acceleration or specialization is defined in the Curriculum Guide. As a matter of fact, the so-called honors course at the Grade 12 level was discontinued several years ago. Moreover, information from the 1981 Mathematics Assessment (Robitaille, 1981) shows that the elective courses in mathematics are taken by very few students. For example, the Geometry 12 course is taken by only 6% of the Grade 12 students.

Another structural artifact of many school systems which affects the curriculum is the existence of external examinations. While examination syllabi are sometimes changed to reinforce content innovations, it would appear that the existence of such examinations impedes curricular change at least as often as it reinforces it. Certainly exit or entry tests generally influence which topics teachers emphasize in their teaching and the instructional strategies employed. While external examinations are usually associated with Europe and many developing countries, the influence on curriculum of tests which are used for screening purposes in the United States--such as the Scholastic Aptitude Test--cannot be discounted. Like the closely related issue of comprehensive versus selective schools, the actual effects of external examinations on school curricula is both an emotive issue and one

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[10]According to two papers recently published by Unesco (Hayter, 1980; Kawaguchi, 1980), for example, qualified students in the United Kingdom and Japan may spend between 10 and 18 hours a week taking mathematics courses at the senior secondary level.

in which further study and analysis seem to be warranted.

A pedagogical situation which appears to be unique to British Columbia is that of semestering or, in some cases, quartering of mathematics courses. In some schools, for example, students take a course such as mathematics during the first semester of the school year, and then take no mathematics course during the second semester. In other parts of Canada and in the United States, this type of semestering of courses may be relatively common at the Grade 12 level, but is rarely seen at the lower grade levels.

Technological Factors

In many places, including Canada, the influence of technological factors on the mathematics curriculum and the teaching of mathematics has been minimal. Results from a survey of mathematics teaching practices in British Columbia (Robitaille and Sherrill, 1977) indicate that little use is made of audio-visual aids and equipment other than the overhead projector. Results from a similar survey conducted in 1981 (Robitaille, 1981b), also show that only a small percentage of teachers who have access to computers actually make use of them in their teaching of mathematics either for instructional or managerial functions. The use of calculators is strongly endorsed by teachers for secondary school students, but not nearly as strongly for elementary school students.

Other countries appear to make more extensive use of various media, for example, in the production of radio and television programs in mathematics for the schools. Heimer (1979) states that, at a conference held in 1967, reports of radio and television programs in mathematics were received from Denmark, Australia, France, Hungary, Ireland, and the United States. He also states that since 1957, the BBC has produced more than 300 such programs, and that the Federal Republic of Germany, Brazil, Japan, the United States, and Hong Kong have "recently mounted important television or radiovision projects in mathematics." (page 222) Of course, without further information, one must be careful not to equate the existence of particular programs or innovations with their widespread use and implementation.

Of all the various technological devices and materials available, the calculator seems likely to have the greatest impact on the schools and on the teaching of mathematics. In some countries their use is virtually universal and students use them at all times, even during examinations. In other countries calculators are rarely found in schools and their impact has yet to be felt.

The situation in British Columbia regarding calculators is somewhere between these two extremes, and some decision regarding their place and availability in the schools will have to be made in the near future. The question of whether it makes sense, economically or educationally, to devote so much time to the development of children's ability to compute when low-priced, powerful calculators are readily available must be addressed. Wheatley (1980), for example, has proposed the following changes to the mathematics curriculum of the elementary school:

- shift from a computationally based curriculum to a conceptually oriented curriculum using the calculator as an instrumental tool, and
- eliminate the teaching of complex computations in the elementary school. (page 37)

Wheatley's second recommendation is the more straightforward of the two. Suggestions to eliminate such computational skills as long division with divisors of more than two digits, and multiplication by multipliers having more than two digits are clear and unambiguous. His first point is more problematic, both in terms of its meaning for curriculum developers and of the form in which such a curriculum would in fact be implemented by teachers. As Roberts (1980) has stated in reviewing the results of research into the use of calculators:

> Although the proposition that calculator usage can have an impact on mathematical concept formation seems reasonable, it is not supported thus far by the empirical data available. In fact, a strong case can be made that this hypothesis has not been adequately tested since few studies made any real attempt to carefully integrate calculator use into the curriculum that would illustrate how calculators can facilitate concept learning. (page 84)

While the phrase, "conceptually oriented curriculum", is open to many, perhaps rather contradictory, interpretations, any forecast of curricular change in this direction arising from the introduction of calculators into the clasroom must be tempered by data such as that reported by teachers in the 1977 British Columbia Mathematics Assessment (Robitaille & Sherrill, 1977). When asked where the major emphasis should be placed in the mathematics curriculum, less than 7% of the elementary school teachers polled favored a greater emphasis on the development of concepts and principles than on computational skills and drill. (page 43)

Given the preeminent postion of computational skills in the curriculum, at least in North America, on the one hand, and the power and availability of calculators on the other, it would appear that some sort of large-scale curricular impact is inevitable. The fate of previous innovations as well as the influence of other factors within the curriculum development process, however, makes it virtually impossible to predict the precise nature of any such change with any high degree of confidence. For those undertaking curriculum revision, an analysis of the uses of calculators and the implications of such use would seem to warrant top priority.

The ever-increasing availability of micro-computers also has important implications for both the content of the school mathematics curriculum as well as the emphasis to be placed on different topics. Ultimately, the presence of such devices in classrooms will have implications for teaching methodology as well.[11]

Summary

The curriculum development process in mathematics is influenced or affected by a number of factors, four of which have been discussed here: sociological, psychological, pedagogical, and technological. In every place where mathematics is taught, different weight is attached to and different concerns dominate each of these factors. This has the ultimate effect of producing different curricula, each of which is unique to the particular place for which it was developed.

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[11]For a utopian image of mathematics learning transformed by computer environments see Papert (1980).

PART IV - THE MATHEMATICS CURRICULUM

Earlier, in Figure 1, the school mathematics curriculum was portrayed as being an outgrowth of the nature of the subject itself, adapted, filtered, and restructured by the curriculum development process. Four factors which influence that process were described and their role in the emergence of a particular mathematics curriculum is portrayed in Figure 2.

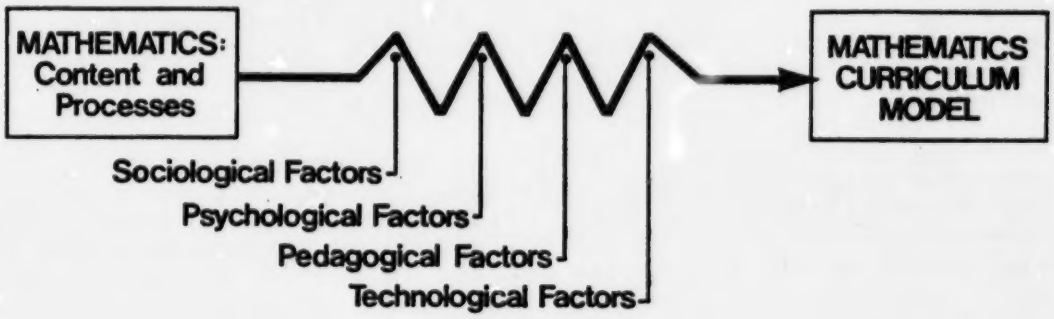

Figure 2. Development of a mathematics curriculum.

One might conceive of several different mathematics curricula, each based on a particular view of the nature of the subject and conditioned by sociological, psychological, pedagogical, and technological factors. These different curricula may be categorized according to the view of the nature and role of mathematics which they espouse. Three such models, or "ideal types" in the terminology of Max Weber (1964), appear to be prevalent today: a model which emphasizes mathematics as the science of abstract structures and their properties, a second which stresses the applications of mathematics to other disciplines, and a third which emphasizes the importance of mathematics in everyday living. These models will be referred to respectively as the Pure Mathematics model, the Applied Mathematics model, and the Basic Mathematics model.

The Pure and Applied Mathematics models are more or less directly derived from the corresponding views of the nature of the discipline. In contrast, the third model has only a

tenuous connection with any view a mathematician might have of the nature or the value of the subject. The Basic Mathematics model is much more a direct product of the influence of factors such as the sociological and pedagogical than a derivative of a particular view of the nature of mathematics.

In practice, no mathematics curriculum would likely be based exclusively on any one of the three models. Thus, the mathematics curriculum for a given place and for a given grade level might advocate placing greater emphasis on one model than on either of the others, but it is unlikely that a curriculum would be constructed on the basis of one of these models to the total exclusion of the other two. For example, a particular curriculum might assign a pre-eminent role to the Pure Mathematics model and yet still include aspects of the Applied Mathematics and Basic Mathematics models. Similarly, it is difficult to imagine an entire mathematics curriculum being built solely around the Basic Mathematics model.

The three models or ideals may be viewed as occupying the vertices of an equilateral triangle. The more a given curriculum derived from one of these three models, the closer that curriculum would lie to the corresponding vertex. As views of the nature of the discipline and the influence of sociological, psychological, pedagogical, and technological factors change, the curriculum for a particular jurisdiction could be seen to move within the interior of the triangle.

To illustrate how closely individual curricula correspond to these models, three examples will be considered: the French, the British, and the North American. The French and British examples illustrate the existence of conflicting views of the importance of the Pure Mathematics model as opposed to the Applied Mathematics one. The North American example is important to this discussion because it appears to be in a somewhat fluid state at the present time as a result of a number of pressures for change that are being exerted both from within and outside the profession.

Due to limitations of space, the discussion of these curricula must be rather brief. Many details have had to be omitted and the resultant portrayal is somewhat over-simplified. In each of the three cases considered, the mathematics curriculum and the issues surrounding it are significantly more complex and less easy to categorize than might be inferred on the basis of the information presented here.

The French Curriculum

French society is more class-structured than North

American society and the French educational system, by its selectivity, remains, despite attempts to democratize it, one of the major agencies propagating that structure (Revuz, 1979). Revuz also points out that mathematics is the major instrument for sorting students into different educational tracks, and he warns that continuation of this practice can do nothing but harm to the future of mathematics in France. He says, for example, "Success in mathematics has become the quasi-unique criterion for the career choices and the selection of pupils." (page 247) And later, "The use of mathematics as a selection instrument within the school system could spell death to mathematical education in France." (page 250)

France has had a long history of excellence in mathematics, but the direction of French mathematics research has changed dramatically in the last hundred years. As Kuhn (1977) points out, French mathematicians such as Laplace, Fourier, and Poisson mathematized physics during the first quarter of the nineteenth century but, for reasons which are unclear, after mid-century the attention of French mathematicians was diverted from the concrete concerns of physics toward the foundations of mathematics, culminating in the voluminous work of Nicolas Bourbaki.[12]

This fundamental re-orientation of mathematical focus has had an impact on the school curriculum. The French mathematician, René Thom, speaking at the second International Congress on Mathematical Education in 1972 said, "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. The modernist tendency is grounded essentially in the formalist conception of mathematics." (Otte, 1979, page 123)

An indication of the direction taken by reform in mathematics education in France may be found in the paper presented by the French mathematician, Jean Dieudonné, at the Royaumont Seminar in 1959 (OEEC, 1961).[13] At that conference, Dieudonné encouraged delegates to consider a number of fundamental re-

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[12]Bourbaki is the pseudonym adopted by a group of mathematicians, mainly French, who produced their first volume in 1939 and their thirty-first in 1965. André Weil and Jean Dieudonné have been among the leaders of the group. The origin of the name itself is obscure, perhaps borrowed from the Franco-Prussian War general, Charles Denis Sauter Bourbaki. (Boyer, 1968)

[13]In the proceedings of that conference, one finds Dieudonné's now-famous remark: "Euclid must go!"

forms to the school mathematics curriculum. He encouraged the adoption of more precise terminology and language, and the replacement of traditional Euclidean geometry by the study of linear algebra and, particularly, of vector spaces. He also urged the adoption of a more rigorous, axiomatic development of school mathematics based upon "constant appeal to intuition" (page 39) (emphasis in the original) and "experimental work", particularly with younger children. In a later paper reviewed by Stevens and Garfunkel (1975), Dieudonné is quoted as saying that, for most teachers, "the only way they can arrive at a reasonably good understanding of mathematics and pass it on to their students will be through a careful presentation of their material, in which definitions, hypotheses, and arguments are precise enough to avoid any misunderstanding." (page 685) In the same vein, André Magnier, Honorary General Inspector of Public Education (France), said "the axiomatic presentation has become--since Hilbert--the key presentation method in mathematics." (page 10)

Many of Dieudonné's recommendations have become hallmarks of what North Americans understand to be the French mathematics curriculum. Similar programs emphasizing mathematics as the study of abstract structures have been developed in Belgium and, for many North Americans, the texts and materials produced in Belgium by Georges Papy served as an introduction to the new approaches being tried in France and other francophone jurisdictions. In the United States, the Comprehensive School Mathematics Program directed by Burt Kaufman has drawn heavily upon Papy's work.

The elementary school mathematics curriculum in Belgium was described recently in a paper by De Bruyn (1980). He lists as major themes of the program, the following five topics: Sets and Relations, Numbers, Geometry, Measurement, and Applied Mathematics. The detailed listing of the content of the program is not broken down by age or grade level, but it is understood that "the ordering is not chronological, but the themes should evolve in symbiosis and the mathematical structures build up themselves progressively in filigree of the five themes." (page 19)

The Sets and Relations theme or strand contains topics such as the properties of equivalence relations, Venn diagrams, quantifiers, logical connectives, and Cartesian products. The Numbers strand includes the concept of infinite sets, integers, rational numbers, properties of operations, and arithmetic. The Geometry strand includes the study of plane and solid figures and their properties as well as the study of projections, symmetries, rotations, and translations. In the Measurement strand are topics such as length, area, volume, mass, time, and temperature. Applied mathematics accounts for approximately 20% of the curriculum and consists of

topics such as equations, statistics, and probability.

In the same volume, a paper by Laumen, Bex, and Nachtergaele (1980) describes the content of the secondary school mathematics curriculum in the following terms:

> "The programs recommend the study of the main ideas concerning sets, relations, and functions; then the study of sets of numbers as structures; in geometry, the study of different sets of transformations and of their basic structure and that of vector spaces of 2 and 3 dimensions.
>
> "The teaching of analysis includes elementary concepts of topology.... Moreover, we hope to reach a unified teaching of mathematics, where the boundaries between algebra, geometry, trigonometry, etc. have been explicitly abolished.
>
> "Because of the introduction of modern concepts and the elimination of topics such as spherical trigonometry, descriptive geometry, and certain areas of arithmetic, we have been able to include in the secondary school curriculum not only an introductory treatment of algebraic structures, but also of integral calculus, statistics, and probability." (page 28) (translated from the French)

Closer to home, the Pure Mathematics model has had an impact on the mathematics curriculum in the Province of Quebéc, as well as in other parts of Canada. In his work at the Université de Sherbrooke, for example, Zoltan Dienes[14] developed materials which emphasized mathematical structures. Moreover, the goals ranked first and second in the list of four basic aims for the elementary mathematics program for francophone schools in Quebéc are:

a. The exploration or acquisition of mathematical concepts, properties, relations, patterns, or structures.

b. A growing familiarity with certain elements (verbal, graphic, or symbolic) of mathema-

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[14]Dienes' materials have also been used to some extent in France, and elsewhere. (Bell, 1975)

> tical language necessary or useful for communication. (McNicol, 1980, page 11) (translated from the French)

Computational skills are ranked third in this list, whereas they are ranked higher in each of the other provinces (McNicol, 1980).

It would be erroneous to assume that, even in France, the Pure Mathematics model has been universally adopted and accepted. Revuz (1979) reveals the existence of a considerable credibility gap regarding the new mathematics programs by saying:

> And there persists among the general public, and within government circles, the idea of an opposition between classical mathematics, austere but useful, of which it is necessary to acquire a "core" at all costs, and modern mathematics, clever and entertaining but in the last resort pointless, which one would put on the list of subjects of "enlightenment" - which might be a tribute to its value as mental training, or a polite way of saying that it is of secondary importance, to be sacrificed when there is a cut in the timetable. (pages 243-244)

There are also indications, from certain spokesmen for the teachers, of dissatisfaction with current programs and practices in the teaching of mathematics. The October 1976 issue of the Bulletin Inter-IREM is devoted to a discussion of this topic under the title "Social Functions of the Teaching of Mathematics".

Granted, then that these ideal curricular types should be applied cautiously, the notion of a distinctive French model for school mathematics seems valid. Howson et al. (1981) offer this characterization:

> The emphasis on logic, orderliness of mind, and clear thinking associated with the French--ce qui n'est pas clair n'est pas francais--characterises their 'modern maths' texts in the same way in which a pragmatic approach does that of their English counterparts. When Burke (Reflections on the French Revolution) remarked that he could find "not one reference whatsoever... to anything moral or anything politic; nothing that relates to the concerns, the actions, the passions, the interests of men," he could well have been

reviewing a French modern mathematics text. (pages 51-52)

The British Curriculum

The traditional British school organization and curriculum have had a tremendous influence on school systems throughout the English-speaking world. In the realm of mathematics, for example, Howson (1979) describes the algebra text written by Hall and Knight as one "on which it could be claimed 'the sun never set' - for it was to be used for decades throughout the British Empire."[15] (page 156) This influence has abated somewhat, but is still very strong in such diverse places as India, Hong Kong, and the West Indies.

The British School system, in contrast to the French, is highly decentralized and "curriculum decisions are normally made at the school level." (Hayter, 1980, page 88) A major constraint on the freedom of the schools to initiate change would appear to be the external examinations which are prepared for secondary school students. Hayter says that "it is an anomalous feature of curriculum planning that, while the system would seem to enable innovation at a local level and provide the possibilty of a mass movement within the system, it more normally appears as a damping medium for the implementation of innovation." (page 88) In particular, he mentions the effect of pressures to resist innovation as including parental and student expectations, teacher expertise, and the existence of external examinations.

The dominant British view of mathematics has always emphasized applied mathematics, and applied mathematics has historically been prominent both in the British mathematics curriculum and in mathematics research. Griffiths and Howson (1974) tell us that "throughout Britain 'Mathematics' in the twentieth century has always included 'Applied Mathematics', and it has always been the aim of schools to stress the applications of the subject." (page 141)

An examination of the material produced by the Nuffield Project and the School Mathematics Project (SMP) will confirm what Griffiths and Howson have said in comparing New Math in Great Britain to that in the U. S. According to Griffiths and Howson (1974), the reason that the first American New Math project, headed by the late Max Beberman at the University of Illinois, was considered to be of little relevance to Britain

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[15]Their trigonometry text, first published in 1893, was still being used in anglophone schools in Québec in 1965.

may have been that " 'modern' mathematics does not consist entirely of abstract algebra in England." (page 141) They also say that, consistent with this British view of mathematics, "great attention was paid to modern industrial applications of mathematics" from the beginnings of curricular projects in the late 1950s. They contrast this with what they view as an almost total absence of any applied orientation in the SMSG materials from the United States or in Papy's Belgian project.

Bryan Thwaites (1972), director of SMP, says:

> It is a general belief in this country that mathematical concepts should be revealed gradually, being introduced at an intuitive level and developed in parallel with other intuitive ideas, so that patterns and logical frameworks slowly emerge....This view of the layout of a school mathematical course is in interesting contrast to, for example, the typical modern American view, in which early formality is thought to be concomitant with understanding, but it is one which has been propounded over many years in this country by such bodies as our Mathematical Association.... Intrinsic in the preceding points, and of fundamental importance to the way in which children learn, is the need at all levels for multitudes of applications of the basic mathematical concepts. (page 241-242) (emphasis added)

At the Fourth International Congress on Mathematical Education (ICME-IV) which was held at the University of California at Berkeley in August 1980, a leaflet entitled SMP News offered the following description of the approach taken in the materials produced by that project:

> The "modern" content, with new topics such as matrices, statistics, and vectors displacing some of the conventional algebra and euclidean geometry, was the most striking feature of the first SMP text-books. However, SMP materials embody an approach to mathematics teaching in which:
>
> - a topic is introduced through a situation or application and the mathematics is developed from that situation;
> - a mathematical idea is developed gradually, and the course returns to it in a helical fashion, to extend and deepen understanding;
> - explanations and justifications are given

for techniques and methods, which are always set in context;
- arbitrary rules are avoided;
- the text is written to be complete in itself, with the intention that the pupil can read it and thus take an active part in his learning.

The fairly obvious and understandable advocacy stance of the above remark must be borne in mind. Only through careful examination and observation of classrooms could the validity of these claims be substantiated. Nonetheless, examination of the contents of the SMP textbooks may offer a useful basis for making comparisons with North American materials.

SMP Books A - H are designed for use with students of average ability (the middle 50% of the population) in the 11-16 year age range. Book D could then be reasonably compared to materials prepared for use in Grade 8 in North America. The Table of Contents of Book D contains the following chapter titles: Group Tables, Rotational Symmetry in 3-D, Order and Punctuation, Similarity and Enlargement, Multiplication and Division of Fractions, Vectors, Multiplication and Division of Directed Numbers, Looking for an Answer, Experimental Probability, Interpretation of Graphs, Number Patterns - Pascal and Fibonacci, Ratio and Percentage, Pythagoras.

Books 1 - 5 are designed for use by students between the ages of 11 and 16 who are in the upper quartile of the ability range. Accordingly, Book 3 might be considered more or less equivalent to the texts approved for use in North America at the Grade 8 level. The contents of Book 3 are as follows: Probability, Isometries, Matrices, Rates of Change, The Circle, Networks, Three-Dimensional Geometry, Linear Programming, Waves, Functions and Equations, Identity and Inverse, Shearing, Statistics, Computers and Programming, Loci and Envelopes.

Another distinctive feature of the British mathematics curriculum has been the extent of teacher involvement in the development of new curricular materials. Howson (1979) says:

> One of the most successful of all projects--judged, faute de mieux, by degree of acceptance and longevity--is the School Mathematics Project of England. This is noteworthy for the fact that not only have all its writing groups consisted almost entirely of practising teachers, but it has consistently used part-time rather than full-time writers. Authors have written in their free time or have been given relief from some of their teaching

> duties to study and write. As a result, they have retained day-to-day contact with the classroom and have been able to try out their ideas there before committing them to paper. (page 144) (emphasis in the original)

The North American Curriculum

Although the title of this section includes the term "North American", it could as easily have referred only to the U. S. A. In the past twenty years, Canadian provinces, with the possible exception of Ontario, have all adopted mathematics textbooks which were either produced in the United States or were produced originally in the U. S. and then "Canadianized".[16] At the present time, there would not appear to be a distinctively Canadian view of the school mathematics curriculum.

Until very recently, it was clear that the dominant view of the nature of mathematics held by mathematicians in the United States was, as in the case of France, one which promoted pure mathematics. Barrett (1979) reports that American mathematics research was dominated by the goals of formalism, and reveals his own biases by saying that "a whole generation of mathematicians labored to abolish their subject by turning it over to the mechanism of axioms."(page 103) Begle (1974) had the following to say about the importance of the formalist school of thought and its eventual application to the school mathematics curriculum:

> "Most mathematics educators extol structure for sound historical reasons. During the early part of the eighteenth century it became clear that further progress in mathematics itself would require that the basic concepts be rethought, clarified, and made more precise. It also became clear that in certain aspects of mathematics, clever and intricate computations were less effective than a careful study of the structure of the mathematical system, the way in which the basic ideas fit together.
>
> "This new point of view toward mathema-

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[16]This means that the texts have been changed so that they use only metric units of measurement and that minor changes have been made in order to make the content more appropriate to Canadian students.

> tics led to a tremendous flowering of mathematical activity. The kind of work being done by creative mathematicians during the early decades of this century was qualitatively different from what was done a century before and was both more powerful and broader in scope.
>
> "During the period between the two World Wars, it became clear that this way of looking at mathematics--first making sure that the basic ideas were clear and understandable and then investigating the ways they fit together--was equally useful in education at the graduate level. By 1940 graduate texts taking this point of view were in the majority, and after World War II the movement spread to the undergraduate program.
>
> "When, in the late fifties, university mathematicians indicated a willingness to assist in the improvement of precollege mathematics, they naturally suggested that the point of view that had proved so successful at higher levels be tried at the secondary school level, and when it seemed to work there, that it be tried at the elementary school level as well." (page 27)

The curriculum development model used in producing many of the major American New Math programs was the Research-Development-Diffusion model. (House, 1979; Howson, 1979) A case history of the application of this paradigm may be found in Wooton (1965) where he describes the early work of SMSG. Briefly, their materials were produced in the following way: a team of mathematicians and teachers would meet for a number of weeks to discuss objectives and write materials. These materials would then be tried out in a number of schools and revised. Finally, the materials were made available for general use. In the case of SMSG and several other projects, financial support was provided mainly by the federal government.

The influence of SMSG on the North American mathematics curriculum has been pervasive, and many textbook series, whether at the elementary or the secondary level, are patterned after their SMSG counterparts. Indeed, many of the authors of contemporary mathematics texts in the United States were, at one time or another, involved in SMSG writing teams.

O'Brien (1973) refers to the important role played by SMSG in describing the content of mathematics at the elemen-

tary school level in North America. He says:

> The goals of present elementary school mathematics programs--derived largely from the publications of the School Mathematics Study Group--involve an understanding of the principles of the rational number system and the interrelatedness of various subsets of the rationals, the use of underlying principles such as associativity and distributivity in support of proficiency in computation and, to a lesser extent, a knowledge of nomenclature and notation in geometry. (pages 258-259)

At the secondary school level, the following quotation from the Preface to one of the SMSG texts, Mathematics for Junior High School, Volume I, provides a brief summary of the flavor of the materials produced by the project:

> Key ideas of junior high school mathematics emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school. (page v)

The two most widely-used mathematics textbooks at the Grade 8 level in British Columbia at the present time are School Mathematics II (Addison-Wesley) and Mathematics II (Ginn and Company). Results from a survey of teachers conducted as part of the 1977 B.C. Mathematics Assessment reveal that about 55% of the teachers of Grade 8 mathematics in the province made use of the former, and an equal number used the latter.[17] (Robitaille and Sherrill, 1977)

Both of these texts owe their origins to SMSG materials. As the authors of Mathematics II state, "the series incorporates many of the recommendations of experimental groups

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[17]The total exceeds 100% because teachers are encouraged to make use of several authorized texts with their mathematics classes.

such as the School Mathematics Study Group, where one of the authors has served as a panelist and both have served as members of writing teams." (page v) They go on to describe the contents of the text as follows:

> Mathematics II is designed to continue the student's work in secondary mathematics, and it offers a sound and thorough preparation for subsequent formal courses in algebra and geometry. Significant attention is directed to the structure of mathematics, to the fundamental ideas underlying the familiar practices and procedures of arithmetic, and to the properties and and relations of algebra and geometry. (page v)

The chapter titles in the text are Integers and Rational Numbers; Exponents and The Pythagorean Property; Decimals and Real Numbers; Constructions and Congruency; Equations and Inequalities; The Coordinate Plane; Formulas and Functions; Prisms and Pyramids; Cylinders, Cones, and Spheres; Probability and Statistics; Similar Figures and Trigonometric Ratios. School Mathematics II contains similar material.

It is difficult to determine the nature of the curriculum model now prevailing in the United States. For a number of reasons, including criticism from mathematicians (e.g. Kline, 1974; Ahlfors and others, 1962) and concerns about purportedly serious declines in standards, there has been a clear move away from much of the New Math content. It is less clear in what direction the curriculum is now heading.[18]

The NACOME[19] report contains a strong rebuttal of criticisms such as those by Kline in his book, Why Johnny Can't Add (1974). The report defends the changes made between 1955 and 1975 by saying that "from a 1975 perspective the principal thrust of change in school mathematics remains fundamentally

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[18]Freeman, Kuhs, Porter, Knappen, Floden, Schmidt, and Schwille (1980) concluded after analyzing three popular Grade 4 textbooks published in 1978 that "all three textbooks reflect the current back to basics and minimal competency trends in their common emphasis on computational skills." (page 2)

[19]NACOME is the acronym for the National Advisory Committee on Mathematical Education which was established by the Conference Board of the Mathematical Sciences in the U. S. A. in 1974.

sound, though actual impact has been modest relative to expectations." (page 21)

The report goes on to discuss current programs and issues. Noting that most New Math programs were directed at the more able students, the report lists the following areas of current concern: "programs for less able students", "minimal mathematical competence for effective citizenship", "interaction of mathematics and its fields of application", and "the impact of new computing technology on traditional priorities and methods in mathematics." (page 23) A similar list of priorities for the 1980s was published recently by the NCTM (1980). They propose that:

- problem solving be the focus of school mathematics in the 1980s;
- basic skills in mathematics be defined to encompass more than computational facility;
- mathematics programs take full advantage of the power of calculators and computers at all grade levels;
- the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;
- more mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accomodate the diverse needs of the student population. (page 1)

It would appear that the NCTM, in referring to "basic skills", is attempting to put the widespread demands for a renewed emphasis on computational skills into a broader perspective. While it would be an exaggeration to say that mathematics instruction in schools has historically meant drill in arithmetic skills, there is no denying the fact that the Basic Mathematics model has held a prominent position in mathematics education in North America ever since elementary education became universal at the turn of the century. At the secondary school level, this model may be seen in the heavy emphasis accorded to the development of manipulative techniques in algebra. This emphasis has been questioned on a number of occasions over the last eighty years by mathematicians, by mathematics educators, and by their professional associations. They have sought, on these occasions, to moderate the stress placed on what might be termed a narrow,

utilitarian, skill-centred, school mathematics curriculum.[20]

At the present time, the mathematics curriculum in North America is undergoing serious scrutiny and fundamental changes will likely be made in the 1980s. O'Brien (1973) summarizes many of the basic questions and issues which must be faced in the near future. He asks:

> What is the usefulness in the life of a child--or in the life of the adult he will become--of an understanding of the structure of the rational number system? What is the usefulness of a "knowledge" of facts, rules, and procedures of number? What is the marketability of computational skill when thousands of error-free long divisions, for example, can be performed by a computer in a few seconds at a cost of a few cents? What is the usefulness in the life of a child or an adult of a "knowledge" of the nomenclature and the notation of elementary geometry? (page 259)

In other words, while it is true that the Basic Mathematics model has been of central importance to the North American mathematics curriculum throughout most of this century, it may well be that the increasing impact of technological factors will necessitate a reconsideration of the amount of emphasis which that model should receive.

Summary

The mathematics curriculum may be viewed in the light of three models or ideal types: the so-called Pure Mathematics model, the Applied Mathematics model, and the Basic Mathematics model. A particular curriculum will typically include aspects of all three models, with one receiving greater emphasis than the other two. That emphasis may vary from one grade level to the next.

In France, the Pure Mathematics model is pre-eminent, and the basic direction and content of their curriculum appears to be fairly firmly established. In Great Britain, the Applied Mathematics model is predominant. In North America, there has been a clear move away from the Pure Mathematics model which prevailed during the New Math era. It is less clear at this

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[20]See, for example, the volume produced by the Progressive Education Association in 1940, Mathematics in General Education.

time which model will be the predominant one in the near future, although some trends are discernible.

There is a certain amount of pressure, for example, to implement a curriculum which emphasizes the Basic Mathematics model. Advocates of the back-to-basics movement stress the importance of computational skills and of preparing students to function as "enlightened consumers in a technological society". Teachers of mathematics, as well as others involved in the field of mathematics education, have reacted to this pressure, not by attempting to belittle the importance of the "basics", but rather by proposing to expand the list of basic skills to include such topics as probability and statistics, and computer literacy.[21]

At the same time, there is a movement to assign higher priority to problem-solving and the applications of mathematics. The NCTM has, in the last few years, published a two-volume text entitled Algebra through Applications (Usiskin, 1979); three yearbooks, Applications in School Mathematics (Sharron, 1979), Problem Solving in School Mathematics (Krulik, 1980), and Teaching Statistics and Probability (Shulte (1981); and a reference text, A Sourcebook of Applications of School Mathematics (Bushaw, 1980). Moreover, the topic of problem-solving was listed by the NCTM (1980) as the number one priority for the teaching of mathematics in the 1980s.

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[21]See, for example, the article entitled "Position Statements on Basic Skills" in the February 1978 issue of The Mathematics Teacher.

PART V - IMPLICATIONS FOR CURRICULUM REVISION

The mathematics curriculum may, for the sake of analysis, be viewed from three different perspectives. Thus, we may distinguish among the curriculum as intended, the curriculum as implemented, and the curriculum as attained. By the Intended Curriculum is meant the curriculum as planned at the national, provincial, or local levels by curriculum committees and consultants, and as codified in curriculum guides. The Implemented Curriculum is the curriculum as contained in the various texts and materials which are selected and approved for use in the schools and as communicated to students by teachers in their classrooms. The Attained Curriculum is the curriculum as learned and assimilated by students. It has been argued elsewhere (Robitaille, 1980), that significant discrepancies exist among these three versions of the curriculum.

In British Columbia, for example, most of the mathematics texts utilized are Canadianized versions of textbooks published in the United States, and, as the discussion earlier indicated, their goals and content have been strongly influenced by those of the SMSG publications. On the other hand, the B. C. Curriculum Guide (Curriculum Development Branch, 1978), the document which sets out the Intended Curriculum, says that:

> Before any formal mathematics can be understood there must be a wealth of manipulative experiences through which concepts and relations are understood at an intuitive level. Mathematics as a discipline, as a formal structure, must be built upon a sound foundation of concrete experiences. Formal study of mathematics as a structured discipline is the function of post-elementary education (page 1).

The guide goes on to say that "at the elementary level it is important [that] the mathematics program include facts, computation, and processes." (page 1)

As for the gap between the Implemented and Attained Curicula, the specific content objectives listed in the curriculum guide for each grade are not obvious outgrowths of the kinds of overall goals contained therein, nor is it clear how mastery of these specific content objectives will result in attainment of those goals.

The list of specific objectives given for each grade in the curriculum guide is lengthy and contains both the old and the new. Topics such as sets, geometry, variables and func-

tions, consumer mathematics, and vectors have been included but all of the customary topics (Roman numerals, long division, factoring, ...) are there as well. The curriculum is crowded with topics to be taught and mastered, and the curriculum guide provides little, if any, specific information to teachers to assist them in determining the appropriate methodology or even the best textbook to use. Indeed, since several texts have been authorized for use at each level, the specific objectives for a given grade do not necessarily coincide very well with any one of those texts.

The present situation, then, is one in which there appear to be important differences between the Intended Curriculum and the Implemented Curriculum. Moreover, data collected recently in connection with a test-development project (Robitaille, Sherrill, and O'Shea, 1980) indicate the existence of possibly serious discrepancies between the Intended Curriculum and the Attained Curriculum. While these data pertain to British Columbia only, this local situation is undoubtedly far from unique. Certainly the lack of clarity in curriculum guidlines extends beyond the provincial borders. Fullan and Park (1981) offer confirmation of this phenomenon from Ontario and go on to offer suggestions for an implementation strategy.

> There is strong evidence which suggests that curriculum guides and other local curriculum materials are not clear about what should change, are not convincing about the need for a change, and do not provide adequate materials. Any strategy for change in this province must support better specification of needed changes, acquisition of materials, and especially distribution and discussion among teachers about "the why and how" of change. Implementation can only come through a combination of defining the need (and teachers must do this through some discussion) and acquiring or developing good materials to address the need. Defining the need involves comparing present practice (student performance and teacher behaviour) with desired practice. The relationship between provincial curriculum guidlines and commercial or local curriculum materials (developed within the system and elsewhere) will have to be examined. As we have stated earlier, one part of the solution is to develop curriculum guidlines and local materials which are more clear about the nature of the change or revision in question. But there is a problem here in that highly specified materials may lead to me-

> chanical implementation in which content is taught, while more fundamental issues relating to teaching approaches and beliefs are neglected. The other part of the solution requires substantial teacher involvement by setting up a process through which the need for a change and its implications for concrete action are developed by and with a group of teachers over time. (pages 41-42)

Regarding the content of the curriculum, Usiskin (1980) argues that if topics such as computer literacy, applications, statistics, and geometric transformations are to be added to the mathematics curriculum of the schools, some existing content will have to be eliminated. One cannot attend mathematics conferences or read professional journals and not reach the conclusion that content relating to mathematical modelling, applications, statistics, computers, calculators, and problem-solving will play an increasingly important role in the mathematics curriculum of the 1980s. Moreover, one cannot help but agree with Usiskin that the hope of the 1960s that "structure, deductive reasoning, and unifying ideas would enable some topics to be dealt with more quickly and more easily" (page 414) and thus not require extensive deletions of traditional material has not been realized. Indeed, such a position now seems more than a trifle naive.

Usiskin is also quite correct in saying that, in the process of curriculum revision, criteria for deciding which content is to be included must be developed in order that new material can be added and, equally important, in order that some traditional content can be excluded. Unfortunately, the situation is extremely complex, and proposed solutions will have to include more than a re-formulation of content-oriented selection criteria.[22] It is essential that if, for instance, problem-solving is to become a major focus of the mathematics curriculum, consideration be given to which views of mathematics and mathematical presentation are consistent with that focus. Additional criteria regarding the selection of materials[23] and the training of teachers must also be consi-

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[22]See, for example, Price and Gawronski (1981) and Dalin (1978).

[23]Comprehensive schemes such as that described by Eraut, Goad, and Smith (1975) may provide useful analytic tools for those charged with the task of curriculum revision. However, the curriculum development and learning theory assumptions which underlie any such scheme must be carefully examined.

dered. As Confrey (1980) has observed:

> Not only does one's understanding of one's discipline affect profoundly what content is selected for presentation, but it also affects how that content is presented. The impact of one's theory(ies) of knowledge is both on the content and method of presentation and the interaction between these two cannot be ignored. (page 21)

Finally, the various aspects of the curriculum development process must be considered if revision is to be meaningfully implemented. Returning to the example of problem-solving, one must consider the linkage between the desired learning outcomes and any commitment to, say, behavioral psychology or behaviorally derived mastery learning programs. In short, the curriculum revision process must seek to identify and make allowances for the various aspects of and relationships within the mathematical and educational systems.

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